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THE AUTOCORRELATION FUNCTION OF STRUCTURAL RESPONSE MEASUREMENTS

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SUMMARY

The autocorrelation function of the response of a multi-degree of freedom system is formally derived and analyzed to determine those factors which significantly influence its characteristics. The form of the autocorrelation function is primarily governed by the joint and cross-joint acceptance function from the viewpoint of coincidence as well as the geometric distribution of the pressure cross-power spectral density. Modal cross-coupling by damping also influences the response for some structures.

Autocorrelation functions of response measurements will generally be complex in nature and will consist of a superposition of damped sine and cosine functions of many frequencies. Narrow band filtering of the response measurements may yield autocorrelation functions possessing single degree of freedom characteristics depending on the spatial homogeneity of the pressure correlation functions.

INTRODUCTION

The response of mechanical structures subjected to random loading has been theoretically investigated from many aspects in recent years. The various theoretical efforts have been primarily directed toward the determination of the power spectral density or the mean square response of the structure due to acoustic-structural coupling. Although generally the end objective of response analysis is the Power Spectral Density, a necessary by-product of digital response analysis is the autocorrelation function.

The autocorrelation function of response of a single degree of freedom system due to a homogeneous external pressure distribution is well known. For a white noise excitation, the autocorrelogram gives directly the damping ratio of the system. Such autocorrelograms have physical meaning in regard to system parameters. Since typical autocorrelograms of the response of missile structures to high intensity, inhomogeneous acoustic pressures do not generally exhibit such single degree of freedom characteristics, the question arises as to how does one physically interpret the autocorrelation function of response of a multi-degree of freedom system.

The calculation of theoretical autocorrelograms involves the determination of the acoustic-structural coupling coefficients (joint acceptance squared) developed by Powell.[3] These coefficients can be evaluated practically only under idealized structural configurations and external pressure distributions. It is not the intent of this analysis to compute such autocorrelograms, but to present the formal derivation of the autocorrelation function of response of a multi-degree of freedom system in an effort to determine the relative effect of the various contributing mechanisms to the form of total response function.

DERIVATION OF THE AUTOCORRELATION FUNCTION OF RESPONSE

In developing the expression for the autocorrelation function of the response of a continuous structure at any arbitrary point to loads which vary randomly in both space and time, the following assumptions are made:

1. The characteristics of each natural mode of the system (i.e., mode shape, frequency, and damping characteristics) are known to within acceptable limits.

2. The total response of the system can adequately be represented by a finite linear sum of the individual modal contributions.

3. The system is lightly damped, which implies that cross-coupling of the modes due to the effects of structural damping can be ignored.

Thus using this normal mode approach, the displacement, $u(\beta, \gamma, t)$ at any point of the structure when it undergoes any arbitrary motion can be represented by

$$u(\beta,\gamma,t) = \sum_{n=1}^{N} q_n(t)\,\phi_n(\beta,\gamma). \qquad (1)$$

The mode shape function $\phi_n(\beta,\gamma)$ describes the shape of the surface of the system when vibrating in the n^{th} natural mode. The coordinates (β,γ) are in general orthogonal surface coordinates and are not meant to be restricted to the Cartesian system. The generalized coordinate, $q_n(t)$, describes the displacement of the surface of the system as a function of time and is measured at the point of maximum deflection when the system is executing motion in the n^{th} natural mode. This modal displacement, $q_n(t)$, is given by the solution of the Lagrange equations,

$$M_n \ddot{q}_n(t) + C_n \dot{q}_n(t) + K_n q_n(t) = L_n(t) \qquad (2)$$

where the quantities M_n, C_n, and K_n are the generalized mass, damping constant, and stiffness constant, respectively, corresponding to the n^{th} natural mode of vibration. $L_n(t)$ is the generalized force for the n^{th} mode and is given by

$$L_n(t) = \int_A \phi_n(\beta,\gamma)\, p(\beta,\gamma,t)\, dA \tag{3}$$

where $p(\beta,\gamma,t)$ is the external pressure acting on the surface of the structure of area A and varying in both space and time.

The solution of (2) can be given in terms of the convolution integral

$$q_n(t) = \frac{1}{K_n} \int_0^{\infty} L_n(t-\tau)\, h_n(\tau)\, d\tau \tag{4}$$

where $h_n(t)$ represents the displacement response of the system in the n^{th} natural mode to a generalized unit impulse of displacement in the n^{th} natural mode. The response function $L_n(t)$ is given for small damping by

$$h_n(t) = \omega_n\, e^{-\xi_n \omega_n t} \sin \omega_n t \tag{5}$$

where $\omega_n = \sqrt{\frac{K_n}{M_n}}$

and

$$\xi_n = \frac{1}{2} C_n (K_n M_n)^{-\frac{1}{2}}.$$

With these fundamental definitions one can now proceed to the calculation of the autocorrelation function for the response of the continuous system.

The autocorrelation function R(s) at any arbitrary point (β_0, γ_0) of the system is defined by:

$$R(s) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} u(\beta_0,\gamma_0,t)\, u(\beta_0,\gamma_0,t+s)\, dt \tag{6}$$

It now remains to substitute the relations (1), (3), (4) and (5) into equation (6) and simplify. Thus substituting (3) into (4)

$$q_n(t) = \frac{1}{K_n} \int_A \phi_n(\beta\gamma) \int_0^{\infty} p(\beta,\gamma,t-\tau)\, h_n(\tau)\, d\tau dA, \tag{7}$$

substituting (7) into (1)

$$u(\beta_0\gamma_0 t) = \sum_{n=1}^{N} \frac{1}{K_n} \phi_n(\beta_0\gamma_0) \int_A \phi_n(\beta\gamma) \int_0^{\omega} p(\beta,\gamma,t-\tau)\, h_n(\tau)\, d\tau dA \tag{8}$$

and substituting (8) into (6) we have upon simplifying

$$R(s) = \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\phi_n(\beta_0\gamma_0)\,\phi_m(\beta_0\gamma_0)}{K_n K_m} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_m(\beta'\gamma') \int_0^{\infty}\int_0^{\infty} h_n(\tau)\, h_m(\tau') \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} p(\beta,\gamma,t-\tau)\, p(\beta',\gamma',t+s-\tau')\, dt \right\} d\tau d\tau' dA dA' \tag{9}$$

Letting $t-\tau = t'$, the bracketed term of (9) becomes

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T-\tau}^{T+\tau} p(\beta,\gamma,t')\, p(\beta',\gamma',t'+s+\tau-\tau')\, dt' = R_{\beta\gamma\beta'\gamma'}(s+\tau-\tau') \tag{10}$$

which can be recognized as being the cross-correlation or space-time correlation of the external pressure with lag time equal to $(s+\tau-\tau')$. This function, as can be seen, is dependent on both τ and τ' as well as on s.

If the external pressure is assumed to represent a stationary random process, then the cross-correlation function, $R_{\beta\gamma\beta'\gamma'}(s+\tau-\tau')$, may be defined in terms of the cross-power spectral density $S_{\beta\gamma\beta'\gamma'}(\omega)$ of the external pressure by the relation

$$R_{\beta\gamma\beta'\gamma'}(s+\tau-\tau') = \int_{-\infty}^{\infty} S_{\beta\gamma\beta'\gamma'}(\omega)\, e^{i\omega(s+\tau-\tau')}\, d\omega \tag{11}$$

The cross-power spectral density function is a complex valued function of frequency and is defined by

$$S_{\beta\gamma\beta'\gamma'}(\omega) = C_{\beta\gamma\beta'\gamma'}(\omega) - i\, Q_{\beta\gamma\beta'\gamma'}(\omega) \tag{12}$$

where $C_{\beta\gamma\beta'\gamma'}(\omega)$ and $Q_{\beta\gamma\beta'\gamma'}(\omega)$ are referred to as the co- and quad-spectrum, respectively, of the cross-power spectral density. Both $C_{\beta\gamma\beta'\gamma'}(\omega)$ and $Q_{\beta\gamma\beta'\gamma'}(\omega)$ are real functions of frequency.

By definition

$$C_{\beta\gamma\beta'\gamma'}(\omega) = C_{\beta\gamma\beta'\gamma'}(-\omega)$$

and $$Q_{\beta\gamma\beta'\gamma'}(\omega) = -Q_{\beta\gamma\beta'\gamma'}(-\omega)$$

Hereafter the subscripts $\beta\gamma\beta'\gamma'$ shall be omitted for convenience. It shall be understood that

$$C(\omega) \equiv C_{\beta\gamma\beta'\gamma'}(\omega)$$

and $$Q(\omega) \equiv Q_{\beta\gamma\beta'\gamma'}(\omega)$$

The cross-correlation function, $R_{\beta\gamma\beta'\gamma'}(s+\tau-\tau')$, is a real quantity. Therefore,

$$R_{\beta\gamma\beta'\gamma'}(s+\tau-\tau') = \int_{-\infty}^{+\infty} \{C(\omega)\cos\omega(s+\tau-\tau') + Q(\omega)\sin\omega(s+\tau-\tau')\}d\omega \tag{13}$$

Substituting this expression into (9) we have for R(s),

$$R(s) = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{\phi_n(\beta_0\gamma_0)\,\phi_m(\beta_0\gamma_0)}{K_n K_m} \int_A\int_{A'} \phi_n(\beta\gamma)\,\phi_m(\beta'\gamma') \int_{-\infty}^{+\infty}$$

$$\left\{ \int_0^{\infty}\int_0^{\infty} h_n(\tau)\,h_m(\tau')\,\{C(\omega)\cos\omega(s+\tau-\tau') + Q(\omega)\sin\omega(s+\tau-\tau')\}d\tau d\tau' \right\} d\omega dA dA' \tag{14}$$

By use of the expression for the unit impulse response function, equation (5), the bracketed integrals may be evaluated to obtain

$$\int_0^\infty\int_0^\infty h_n(\tau)h_m(\tau')\ \{C(\omega)\cos\omega(s+\tau-\tau') + Q(\omega)\ \sin\omega(s+\tau-\tau')\}d\tau d\tau' =$$

$$\omega_m{}^2\omega_n{}^2\{([\xi_m{}^2+1]\omega_m{}^2-\omega^2)([\xi_n{}^2+1]\omega_n{}^2-\omega)+4\omega^2\xi_m\xi_n\ \omega_n\omega_m\}$$

$$\{C(\omega)\ \cos\omega\ s+Q(\omega)\ \sin\omega\ s\}\{([\xi_m{}^2+1]\omega_m{}^2-\omega^2)^2+4\omega^2\xi_m{}^2\omega_m{}^2\}^{-1} \tag{15}$$

$$\{([\xi_m{}^2+1]\omega_n{}^2-\omega^2)^2+4\omega^2\xi_n{}^2\omega_n{}^2\}^{-1}$$

Denoting the right side of (15) by $F_{mn}(\omega)$, equation (14) becomes

$$R(s) = \sum_{n=1}^{N}\sum_{m=1}^{N}\frac{\phi_n(\beta_0\gamma_0)\,\phi_m(\beta_0\gamma_0)}{K_n K_m}\int_A\int_{A'}\phi_n(\beta\gamma)\,\phi_m(\beta'\gamma')\int_{-\infty}^{+\infty}F_{mn}(\omega)\,d\omega dA dA' \tag{16}$$

For suitably restricted functions, $F_{mn}(\omega)$, the integral over frequency may be evaluated by contour integration. For n=m

$$\int_{-\infty}^{+\infty}F_{nn}(\omega)\,d\omega = f_{n_1}C(\omega_n) + f_{n_2}Q(\omega_n) \tag{17}$$

where $f_{n_1}(s) = \dfrac{\pi e^{-s\xi_n\omega_n}\ \omega_n}{2\xi_n}\{\cos\omega_n s+\xi_n\sin\omega_n s\}$

$$f_{n_2}(s) = \frac{\pi e^{-s\xi_n\omega_n}\ \omega_n}{2\xi_n}\{\sin\omega_n s-\xi_n\cos\omega_n s\}$$

and $C(\omega_n)$ and $Q(\omega_n)$ are the co- and quad-spectrum of the pressure cross-power spectral density evaluated at the resonant frequency, ω_n, of the structure. These functions are also referred to as spatial correlation functions.

For $n \neq m$

$$\int_{-\infty}^{+\infty} F_{nm}(\omega)(d\omega) = \frac{f_{nm_1}}{f_{nm_5}} C(\omega_n) + \frac{f_{nm_2}}{f_{nm_5}} Q(\omega_n) + \frac{f_{nm_3}}{f_{nm_5}} C(\omega_m) + \frac{f_{nm_4}}{f_{nm_5}} Q(\omega_m) \qquad (18)$$

where

$$f_{nm_1} = \pi\omega_n\omega_m{}^2 e^{-\xi_n\omega_n s} \{ [-(\omega_n{}^2-\omega_m{}^2) + (\omega_n\xi_n+\omega_m\xi_m)^2] \sin\omega_n s$$

$$+ [2\omega_n(\omega_n\xi_n+\omega_m\xi_m)] \cos\omega_n s\}$$

$$f_{nm_2} = \pi\omega_n\omega_m{}^2 e^{-\xi_n\omega_n s} \{ [(\omega_n{}^2-\omega_m{}^2) - (\omega_n\xi_n+\omega_m\xi_m)^2] \cos\omega_n s$$

$$+ [2\omega_n(\omega_n\xi_n+\omega_m\xi_m)] \sin\omega_n s\}$$

$$f_{nm_3} = \pi\omega_n{}^2\omega_m e^{-\xi_m\omega_m s} \{ [(\omega_n{}^2-\omega_m{}^2) + (\omega_n\xi_n+\omega_m\xi_m)^2] \sin\omega_m s$$

$$+ [2\omega_m(\omega_n\xi_n+\omega_m\xi_m)] \cos\omega_m s\}$$

$$f_{nm_4} = \pi\omega_n{}^2\omega_m e^{-\xi_m\omega_m s} \{ -[(\omega_n{}^2-\omega_m{}^2) + (\omega_m\xi_m+\omega_m\xi_n)^2] \cos\omega_m s$$

$$+ [2\omega_m(\omega_n\xi_n+\omega_m\xi_m)] \sin\omega_m s\}$$

$$f_{nm_5} = [(\omega_n{}^2-\omega_m{}^2)^2 + 2(\omega_n{}^2+\omega_m{}^2)(\omega_n\xi_n+\omega_m\xi_m) + (\omega_n\xi_n+\omega_m\xi_m)^4]^{-1}$$

Substituting (17) and (18) into equation (16)

$$R(s) = \sum_{n=1}^{N} \frac{\phi_n^{2}(\beta_0\gamma_0)}{K_n^{2}} \left\{ f_{n_1} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_n(\beta'\gamma')\,C(\omega_n)\,dAdA' \right.$$

$$\left. + f_{n_2} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_n(\beta'\gamma')\,Q(\omega_n) dAdA' \right\} + 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \frac{\phi_n(\beta_0\gamma_0)\,\phi_m(\beta_0\gamma_0)}{K_n K_m} \cdot$$

$$\left\{ \frac{f_{nm_1}}{f_{nm_5}} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_m(\beta'\gamma')\,C(\omega_n)\,dAdA' + \right.$$

$$\frac{f_{nm_2}}{f_{nm_5}} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_m(\beta'\gamma')\,Q(\omega_n)\,dAdA' + \tag{19}$$

$$\frac{f_{nm_3}}{f_{nm_5}} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_m(\beta'\gamma')\,C(\omega_m)\,dAdA' +$$

$$\left. \frac{f_{nm_4}}{f_{nm_5}} \int_A \int_{A'} \phi_n(\beta\gamma)\,\phi_m(\beta'\gamma')\,Q(\omega_m)\,dAdA' \right\}$$

The integrals of (19) are analogous to the joint and cross-joint acceptance defined by Powell, with the exception that they have not been reduced to dimensionless form by dividing by the square of the area and the power spectrum of the pressure at the point $(\beta\gamma)$.

The expression (19) is the formal autocorrelation function for an arbitrary structure subjected to random loading. Although the expression appears to be quite cumbersome, several simplifying assumptions based on the geometric features of the assumed pressure distribution and on the relative magnitudes of the contributing terms can reduce this expression to a tractable form.

GEOMETRIC PROPERTIES OF THE ACCEPTANCE INTEGRALS

The co- and quad-spectra, $C(\omega_n)$ and $Q(\omega_n)$ respectively, are functions of the spatial coordinates as well as functions of frequency and spatial separation of the measurement points. Some general statements can be made as to the significance of these functions under certain conditions on the spatial homogeneity of the pressure distribution.

Consider the integral

$$\int_A \int_{A'} \phi_n(\beta,\gamma)\,\phi_m(\beta',\gamma')\,P_n(\beta,\gamma,\beta-\beta',\gamma-\gamma')\,dA\,dA' \tag{20}$$

If the surface area is bounded, $[(0 \le \beta \le a),\ (0 \le \gamma \le b)]$, then (20) can be rewritten as

$$\begin{aligned}&\int_0^a \int_0^b \phi_n(\beta\gamma) \left\{ \int_0^\beta \int_0^\gamma \phi_m(\beta'\gamma')\,P_n(\beta,\gamma,\beta-\beta',\gamma-\gamma')\,d\beta' d\gamma' \right\} d\beta d\gamma\ + \\ &\quad \int_0^a \int_0^b \phi_n(\beta,\gamma) \left\{ \int_\beta^a \int_\gamma^b \phi_m(\beta'\gamma')\,P_n(\beta,\gamma,\beta-\beta',\gamma-\gamma')\,d\beta' d\gamma' \right\} d\beta d\gamma\end{aligned} \tag{21}$$

Letting

$$\beta = a-\beta'' \qquad \gamma = b-\gamma''$$

$$\beta' = a-\beta''' \qquad \gamma' = b-\gamma'''$$

the second term of (21) becomes

$$\int_0^a \int_0^b \phi_n(a-\beta'',b-\gamma'') \int_0^{\beta''} \int_0^{\gamma''} \phi_m(a-\beta''',b-\gamma''')\,P_n(a-\beta'',b-\gamma'',\beta'''-\beta'',\gamma'''-\gamma'')\,d\beta''' d\gamma''' d\beta'' d\gamma''$$

or

$$\int_0^a \int_0^b \phi_n(a-\beta,b-\gamma) \int_0^\beta \int_0^\gamma \phi_m(a-\beta',b-\gamma')\,P_n(a-\beta,b-\gamma,\beta'-\beta,\gamma'-\gamma)\,d\beta' d\gamma' d\beta d\gamma \tag{22}$$

Therefore (21) reduces to

$$\int_0^a \int_0^b \int_0^\beta \int_0^\gamma [\phi_n(\beta,\gamma)\, \phi_m(\beta',\gamma')\, P_n(\beta,\gamma,\beta-\beta',\gamma-\gamma') +$$

$$+ \phi_n(a-\beta, b-\gamma)\, \phi_m(a-\beta', b-\gamma')\, P_n(a-\beta, b-\gamma, \beta'-\beta, \gamma'-\gamma)]\, d\beta' d\gamma' d\beta d\gamma \tag{23}$$

Generally, the mode shape function $\phi_k(x_1x_2)$ can be expressed as a product of spatial independent modes

$$\phi_k(x_1x_2) = \phi_i(x_i) \cdot \phi_j(x_2) \tag{24}$$

The sub-scripts i and j shall be defined to be integral multiples of (1/2). Also $\phi_\ell(x_k)$ shall be defined to have the property that

$$\phi_\ell(x_k) = \begin{cases} +\phi_\ell(c-x_k) & \ell \text{ is a half-integer} \\ & \text{if} \\ -\phi_\ell(c-x_k) & \ell \text{ is an integer} \end{cases} \tag{25}$$

and if $(0 \leq x_k \leq c)$, that is if c is the "length" of the bounded structure in the x_k direction. In this notation the subscript ℓ denotes the number of modal wavelengths in the x_k direction.

Consider the special case where

$$P_n(\beta,\gamma,\beta-\beta',\gamma-\gamma') = P_n(a-\beta, b-\gamma, \beta-\beta', \gamma-\gamma') \tag{26}$$

that is, if the point (β,γ) were projected onto a plane tangent to the structure at the poing (0, 0) and rotated 180 degrees and then was reprojected back onto the structural surface, P_n at the point (β,γ) and at its rotated image $(a-\beta, b-\gamma)$ would be the same. P_n actually represents either the co- or the quad-spatial correlation functions evaluated at the frequency ω_n. These functions have the additional property that

$$C(\omega_n,\beta,\gamma,\beta-\beta',\gamma-\gamma') = C(\omega_n,\beta,\gamma,\beta'-\beta,\gamma'-\gamma)$$

and

$$Q(\omega_n,\beta,\gamma,\beta-\beta',\gamma-\gamma') = -Q(\omega_n,\beta,\gamma,\beta'-\beta,\gamma'-\gamma) \tag{27}$$

Representing the product $\phi_n(\beta,\gamma)\,\phi_m(\beta',\gamma')$ as

$$\phi_n(\beta,\gamma)\,\phi_m(\beta',\gamma') = \phi_r(\beta)\,\phi_s(\gamma)\,\phi_p(\beta')\phi_q(\gamma')$$

then if $P_n = C(\omega_n)$

then the integral (23) will be identically zero if $(r+s+p+q)$ is an integer. Whereas if

$$P_n = Q(\omega_n)$$

then the integral (23) will be identically zero when $(r+s+q+p)$ is a half-integer.

Therefore, under the assumed conditions (26), the co- and quad-spectra of the pressure cross-power spectral density cannot contribute to the same structural mode pairs. This is a significant result in that under the assumed conditions, which includes the special case of homogeneous turbulence, exactly one-half of the terms of equation (19) are identically zero. In particular, for n=m, the second term, which is one of the most important contributors to the autocorrelation function, vanishes.

REDUCTION BY MAGNITUDE

In the case where the modal damping coefficients are small in comparison to unity, equation (19) can be considerably reduced to a practical form. In reference to (17), for small damping

$$f_{n_1}(s) \approx \frac{\pi\, e^{-s\xi_n\omega_n}\,\omega_n}{2\xi_n} \cos \omega_n s$$

and (28)

$$f_{n_2}(s) \approx \frac{\pi\, e^{-s\xi_n\omega_n}\,\omega_n}{2\xi_n} \sin \omega_n s$$

If the case where $\omega_n >> \omega_m$ is considered, then the remaining coefficients can be approximated by

$$\frac{f_{nm_1}}{f_{nm_5}} \approx -\pi \frac{\omega_m^{\;2}}{\omega_n} e^{-\xi_n \omega_n s} \sin \omega_n s$$

$$\frac{f_{nm_2}}{f_{nm_5}} \approx \pi \frac{\omega_m^{\;2}}{\omega_n} e^{-\xi_n \omega_n s} \cos \omega_n s$$

$$\frac{f_{nm_3}}{f_{nm_5}} \approx \pi\, \omega_m\, e^{-\xi_m \omega_m s} \sin \omega_m s \qquad (29)$$

and

$$\frac{f_{nm_4}}{f_{nm_5}} \approx -\pi\, \omega_m\, e^{-\xi_m \omega_m s} \cos \omega_m s$$

If the cross and cross-joint acceptance integrals of equation (19) are of the same order of magnitude then the cross-terms can clearly be neglected by virtue of the smallness of the coefficients (29) in comparison to those of (28). On the other hand, if $\omega_n \approx \omega_m$, that is when two modal frequencies lie very close to each other such that

$$(\omega_n^{\;2}-\omega_m^{\;2})^2 << 2(\omega_n^{\;2}+\omega_m^{\;2})\,(\omega_n\xi_n+\omega_m\xi_m) \qquad (29)$$

then

$$\frac{f_{nm_1}}{f_{nm_5}} \approx \frac{4\pi\omega_n^{\;2}}{6} e^{-\xi_n \omega_n s} \cos \omega_n s$$

$$\frac{f_{nm_2}}{f_{nm_5}} \approx \frac{4\pi\omega_n^{\;2}}{6} e^{-\xi_n \omega_n s} \sin \omega_n s$$

$$\frac{f_{nm_3}}{f_{nm_5}} \approx \frac{4\pi\omega_m^{\;2}}{6} e^{-\xi_m \omega_m s} \cos \omega_m s \qquad (30)$$

$$\frac{f_{nm_4}}{f_{nm_5}} \approx \frac{4\pi\omega_m^{\;2}}{6} e^{-\xi_m \omega_m s} \sin \omega_m s$$

The magnitudes of the coefficients are extremely larger than those of (28) for high frequencies. In this case the cross-terms may predominate depending on the relative magnitude of the joint and cross-joint acceptance integrals. This condition, (29), seldom occurs in the analysis of plates, but it is a common event in the analysis of shells.

For plate analysis, it is generally assumed that the cross-terms of (19) may be neglected by virtue of the smallness of the coefficients. In this case the autocorrelation function is represented by

$$R(s) = \sum_{n=1}^{N} \frac{\phi_n^2(\beta_0\gamma_0)}{K_n^2} \frac{\pi e^{-s\xi_n\omega_n}\omega_n}{2\xi_n}$$

$$\left\{ \cos \omega_n s \iint_{A\,A'} \phi_n(\beta\gamma)\,\phi_n(\beta'\gamma')\,C(\omega_n)\,dA\,dA' + \sin \omega_n s \iint_{A\,A'} \phi_n(\beta\gamma) \right. \tag{31}$$

$$\left. \phi_n(\beta'\gamma')\,Q(\omega_n)\,dA\,dA' \right\}$$

Powell has shown that the inclusion of the cross-terms causes the calculated response of a continuous structure to be of significance mainly in the region of the most intense excitation, and to be small outside this region. Neglection of these cross-terms causes the response to be large over the whole vibrating structure, irrespective of the degree of concentration of the excitation.

If in addition to neglecting the cross-terms, the co- and quad-spatial correlations functions satisfy (26) then the joint acceptance integrals involving the quad-spatial correlation functions will be identically zero. In such a case the autocorrelation function is represented by a linear superposition of damped cosine functions which decay exponentially with lag time. The decay coefficients of the individual modal decay curves yields directly the modal damping coefficient.

The neglection of the cross-modal contributions implies that the various structural modes are statistically independent. This situation is generally not the case; for a particular frequency component of the pressure fluctuations can excite vibrations in many modes. Conversely, many frequency components contribute to the excitation of a particular mode. The contribution of this mode to the total correlation function will consist of a damped sine and cosine function whose decay coefficient will express the actual damping ratio associated with that mode, and whose amplitude will be a linear sum of the contributions from all natural frequency components. Thus if a single modal contribution to the total autocorrelation function could be isolated, such as by means of a narrow band filter operation on the response measurements, the resulting correlogram could be expressed as

$$R_n(s) = [A_n \cos \omega_n s + B_n \sin \omega_n s] e^{-\xi_n \omega_n s} \tag{32}$$

where A_n and B_n would express the contributions from all frequencies. The total autocorrelation function would be a linear sum of all the frequency components

$$R(s) = \sum_{n=1}^{\infty} R_n(s) \tag{33}$$

JOINT AND CROSS-JOINT ACCEPTANCE INTEGRALS

The magnitude of both A_n and B_n depends very strongly on the magnitude of the joint and cross-joint acceptance integrals. These integrals can be evaluated practically only under very restricted conditions.

In any given analysis the major uncertainty lies in the description of the pressure distribution on the surface of the structure. Generally, the spatial correlation functions are taken to be spatially separable and of the form

$$C(\omega_n) = C_n e^{-a_n \left|\beta-\beta'\right|} \cos a'_n(\beta-\beta') \, e^{-b_n \left|\gamma-\gamma'\right|} \cos b'_n(\gamma-\gamma')$$

and (34)

$$Q(\omega_n) = Q_n e^{-c_n \left|\beta-\beta'\right|} \sin c'_n(\beta-\beta') \, e^{-d_n \left|\gamma-\gamma'\right|} \sin d'_n(\gamma-\gamma')$$

where a_n, a'_n, b_n, b'_n, c_n, c'_n, d_n and d'_n are functions of the spatial coordinates. The form of these functions (34) has been justified to some extent by experiments. The experimental determination of the parameters requires numerous closely-spaced measurements. Unfortunately, due to the lack of adequate instrumentation, such determinations have not as yet been fully exploited. For the purpose of calculation, these parameters are generally assumed to be constant over the structural surface. This condition corresponds to the case of a homogeneous pressure distribution which satisfies the condition of equation (26). Even for this condition, the calculation of the acceptance integral is extremely cumbersome.

Bozich[1] presents calculations of the joint acceptance integral for a variety of plate configurations, which show the relative selectivity of the plate to the pressure wavelength. He shows that when the pressure correlation lengths are small compared with plate dimensions, the joint acceptance function becomes essentially independent of the mode. In such a case the calculations become extremely simplified.

MEAN SQUARE RESPONSE

The mean square response is obtained by setting the lag time equal to zero. This operation considerably simplifies the coefficients $f_{mn}(s)$ defined by (17) and (18). However, one is still faced with the difficulty of evaluating the joint and cross-joint acceptance integrals. For separable cross-correlation functions, Nash[2] has calculated the response of elastic plates to distribute random pressures. Additional simplification can be made by considering the mean square space average of the response as treated by White[5].

CONCLUSIONS

The determination of the response of practical structures by means of the normal mode approach requires detailed knowledge of both the free vibration response of the structure as well as the characteristics of the applied pressure field. Generally, this information is not available and considerable simplying assumptions must be made for the purpose of calculation. Upon examination of the complexity of the autocorrelation function in detail, it becomes obvious that the multiplicity of the errors of uncertainty of both the dynamic structural quantities and the fluctuating pressure field characteristics in physical situations prevents the attainment of reliable calculated values. Moreover, even if the structural and field characteristics were accurately known, the evaluation of the acceptance integrals would present enormous difficulties even by numerical evaluation by high speed digital computers.

The essential value in the derivation of the response of practical structures to fluctuating pressure distributions by means of the normal mode approach therefore lies in providing the means for understanding those mechanisms which significantly influence the response quantities.

This analysis shows that the form of the autocorrelation function of response is governed primarily by the modal damping coefficient, ξ_n, and the relative modal acceptance of the acoustic energy. The degree of energy acceptance will be a maximum when the acoustic wavelength equals the elastic wavelength of the structure and when the corresponding acoustic frequency equals the resonant frequency. Thus the magnitude of the acoustic energy at a particular frequency alone does not govern the actuality of response. The acoustic and elastic wavelength must also be similar. This condition is referred to as coincidence.

In order to obtain an autocorrelation function whose characteristics resemble a pure damped cosine function the spatial correlation functions, $C(\omega_n)$ and $Q(\omega_n)$, must satisfy the condition (26), the modal damping coefficient must be such that (29) is satisfied, and the condition of coincidence must be satisfied only for the acoustic energy of frequency (ω_n). For these conditions to be satisfied simultaneously in physical situations would be a rare occasion. Thus it appears that autocorrelation functions of response measurement will generally be complex in nature and will consist of a superposition of damped sine and cosine functions of many frequencies. Narrow band filtering of the response measurements may yield autocorrelation functions possessing single degree of freedom characteristics depending on the spatial homogeneity of the pressure correlation functions.

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